Paper II – Mathematics

36. What is the cardinality of the empty set?
- **A.** 1
- **B.** 0
- **C.** -1
- **D.** Undefined

37. Which of the following is a valid representation of an empty set?
- **A.** {0}
- **B.** { }
- **C.** 0
- **D.** {∅}

38. The series $\sum ((-1)^n / n)$ is:
- **A.** Absolutely convergent.
- **B.** Conditionally convergent.
- **C.** Divergent.
- **D.** None of the above

39. Which of the following is an equivalent statement to the Bolzano-Weierstrass theorem (for sets)?
- **A.** Every bounded infinite set has a limit point.
- **B.** Every bounded infinite set is closed.
- **C.** Every bounded infinite set is open.
- **D.** Every bounded infinite set is compact.

40. Which of the following functions is uniformly continuous on the interval (0, 1)?
- **A.** $f(x) = 1/x$
- **B.** $f(x) = \sin(1/x)$
- **C.** $f(x) = x^2$
- **D.** $f(x) = \sin(x)$

41. A sequence of functions $\{f_n(x)\}$ is said to be uniformly convergent to a function $f(x)$ on a set S if:
- **A.** For every $\varepsilon > 0$, there exists an N such that for all $n > N$ and for all x in S, $|f_n(x) - f(x)| < \varepsilon$.
- **B.** For every x in S, there exists an N such that for all $n > N$, $|f_n(x) - f(x)| < \varepsilon$.
- **C.** For every $\varepsilon > 0$ and for all x in S, there exists an N such that for all $n > N$, $|f_n(x) - f(x)| < \varepsilon$.
- **D.** For every $\varepsilon > 0$, there exists an N such that for all $n > N$, $|f_n(x) - f(x)| < \varepsilon/2$.

42. Which of the following statements is true regarding the Riemann integrability of a function?
- **A.** A bounded function is always Riemann integrable.
- **B.** A function with a finite number of discontinuities is always Riemannintegrable.
- **C.** A continuous function on a closed interval is always Riemann integrable.
- **D.** A function with an infinite number of discontinuities is never Riemann integrable.

43. What does it mean for an improper integral to "converge"?
- **A.** The limit of the integral exists and is a finite number.
- **B.** The limit of the integral is infinity or does not exist.
- **C.** The integral is undefined.
- **D.** The integral is easy to solve.

44. Which of the following statements is TRUE about functions of bounded variation?
- **A.** A continuous function is always of bounded variation.
- **B.** A function of bounded variation is always continuous.
- **C.** A monotonic function is always of bounded variation.
- **D.** A function of bounded variation is always monotonic.

45. What does the directional derivative of a function measure?
- **A.** The rate of change of the function with respect to time.
- **B.** The rate of change of the function in a specific direction.
- **C.** The maximum rate of change of the function.
- **D.** The minimum rate of change of the function.

46. Which of the following is a metric on a set X?
- **A.** $d(x, y) = |x - y|$ for x, y in X
- **B.** $d(x, y) = (x - y)^2$ for x, y in X
- **C.** $d(x, y) = |x - y| + 1$ for x, y in X
- **D.** $d(x, y) = 1$ if $x \neq y$ and 0 if $x = y$

47. Which of the following statements is true about compact sets?
- **A.** Every compact set is closed and bounded.
- **B.** Every closed and bounded set is compact.
- **C.** A set is compact if and only if it is closed and bounded.
- **D.** A set is compact if and only if it has a finite subcover.

48. What is the condition for two vectors u and v in an inner product space to be orthogonal?
- **A.** $||u + v|| = ||u|| + ||v||$
- **B.** $(u, v) = 0$
- **C.** $||u|| = ||v||$
- **D.** $(u, v) > 0$

49. What is the radius of convergence of a power series?
- **A.** The value of x where the series converges
- **B.** The distance from the center of the series to the edge of the interval of convergence
- **C.** The number of terms in the series
- **D.** The sum of the series

50. If $f(z) = u + iv$ is an analytic function, then which of the following is true?
- **A.** u and v are both harmonic functions.
- **B.** u is harmonic, but v may not be.
- **C.** v is harmonic, but u may not be.
- **D.** Neither u nor v is harmonic.

51. The Cauchy's Residue Theorem is a powerful tool for evaluating:
- **A.** Definite integrals of real-valued functions.
- **B.** Indefinite integrals of real-valued functions.
- **C.** Definite integrals of complex-valued functions.
- **D.** Definite integrals of both real and complex-valued functions.

52. What does the Fundamental Theorem of Arithmetic state?
- **A.** Every composite number can be uniquely expressed as a product of primes.
- **B.** Every prime number can be uniquely expressed as a product of composite numbers.
- **C.** Every integer greater than 1 is either prime or can be written as a unique product of prime numbers.
- **D.** Every even number can be expressed as a sum of two prime numbers.

53. Which of the following statements is true about a normal subgroup?
- **A.** A normal subgroup is always a cyclic group.
- **B.** A normal subgroup is always a subgroup of index 2.
- **C.** A normal subgroup is always a subgroup of index 3.
- **D.** A normal subgroup N of a group G satisfies $gNg^{-1} = N$ for all g in G.

54. If H is a normal subgroup of G, which of the following is always true?
- **A.** H is a cyclic group.
- **B.** H is an abelian group.
- **C.** The left cosets of H in G are the same as the right cosets of H in G.
- **D.** H is a simple group.

55. Which of the following statements is true about prime ideals in a commutative ring with identity?
- **A.** Every prime ideal is maximal.
- **B.** Every maximal ideal is prime.
- **C.** A prime ideal is never maximal.
- **D.** A prime ideal is always a zero ideal.

56. Which of the following is a polynomial ring?
- **A.** The set of all real numbers with standard addition and multiplication.
- **B.** The set of all integers with standard addition and multiplication.
- **C.** The set of all polynomials with real coefficients and standard addition and multiplication.
- **D.** The set of all rational numbers with standard addition and multiplication.

57. Which of the following is true about the product topology?
- **A.** It is the finest topology on the product space.
- **B.** It is the coarsest topology that makes all projection maps continuous.
- **C.** It is the same as the box topology.
- **D.** It is not a topology.

58. What is the purpose of initial conditions in an initial value problem?
- **A.** To determine the type of differential equation.
- **B.** To specify the value of the independent variable.
- **C.** To select a particular solution from the family of solutions.
- **D.** To define the domain of the problem.

59. If $\varphi(x)$ is a solution to a homogeneous linear ODE, then which of the following is also a solution?
- **A.** $2\varphi(x)$
- **B.** $\varphi(x) + 1$
- **C.** $\varphi(x)^2$
- **D.** $\varphi(x) - 1$

60. What is the main advantage of using the method of variation of parameters over the method of undetermined coefficients?
- **A.** It is simpler to apply in most cases.
- **B.** It can be used to solve any linear non-homogeneous differential equation.
- **C.** It requires fewer calculations.
- **D.** It always gives a simpler form of the particular solution.

61. Lagrange's auxiliary equations for the PDE Pp + Qq = R are:
- **A.** dx/P = dy/Q = dz/R
- **B.** dP/dx = dQ/dy = dR/dz
- **C.** P dx + Q dy + R dz = 0
- **D.** None of the above

62. Charpit's method is applicable to:
- **A.** Linear partial differential equations
- **B.** Quasi-linear partial differential equations
- **C.** Non-linear partial differential equations
- **D.** All types of partial differential equations

63. Charpit's auxiliary equations are:
- **A.** Derived from the given PDE and the auxiliary equation.
- **B.** Always linear in p and q.
- **C.** Used to find the complete integral directly.
- **D.** Only applicable to linear PDEs.

64. Which of the following is a fundamental assumption when using the method of separation of variables to solve the Laplace equation?
- **A.** The solution is a constant function.
- **B.** The solution can be expressed as a product of functions, each depending on only one variable.
- **C.** The solution is a linear combination of trigonometric functions.
- **D.** The solution is a polynomial function.

65. The Newton-Raphson method is used to:
- **A.** Find the area under a curve.
- **B.** Solve a system of linear equations.
- **C.** Find the roots of a non-linear equation.
- **D.** Approximate definite integrals.

66. A function is of bounded variation if:
- **A.** It is continuous
- **B.** Its total variation is finite
- **C.** Its derivative is bounded
- **D.** It is differentiable

67. The kernel K(x, t) of a linear integral equation is:
- **A.** A function of 'x' only
- **B.** A function of 't' only
- **C.** A function of both 'x' and 't'
- **D.** A constant

68. What is the sum of all probabilities in a discrete probability distribution?
- **A.** 0
- **B.** 1
- **C.** 0.5
- **D.** It varies depending on the distribution

69. Bayes' Theorem is a formula that calculates:
- **A.** The probability of an event occurring.
- **B.** The probability of an event given that another event has already occurred.
- **C.** The probability of an event given that another event has already occurred, taking into account prior knowledge.
- **D.** The intersection of two events.

70. Which of the following represents a sample space for tossing a coin twice?
- **A.** {H, T}
- **B.** {HH, TT}
- **C.** {HH, HT, TH, TT}
- **D.** (D) {H, T, HH}